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Convergence and stability estimates in difference setting for time-fractional parabolic equations with functional delay

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Abstract

A class of one-dimensional time-fractional parabolic differential equations with delay effects of functional type in the time component is numerically investigated in this work. To that end, a compact difference scheme is constructed for the numerical solution of those equations based on the idea of separating the current state and the prehistory function. In these terms, the prehistory function is approximated by means of an appropriate interpolation–extrapolation operator. A discrete form of the fractional Gronwall inequality is employed to provide an optimal error estimate. The existence and uniqueness of the numerical solutions, the order of approximation error for the constructed scheme, the stability and the order of convergence are mathematically investigated in this work.

KEYWORDS

fractional parabolic differential equations, functional delay, discrete fractional Gronwall-type inequality, compact difference method, convergence and stability

1 | INTRODUCTION

The L_1-type schemes have a wide range of applicability in solving differential equations of fractional order in time [1]. Most reports on L_1-type methods refer to the efficiency analysis of numerical schemes for linear fractional-order problems [2–4]. A few of those works have discussed the stability and the convergence of L_1-type schemes for nonlinear time-fractional differential equations. That is the case for articles like [5, 6]. However, the results were controlled by the locality in time in those cases. Recently, Li et al. [7, 8] developed a fractional Gronwall-type inequality in order to overcome those difficulties.

Using that inequality, the numerical analysis of L_1-type schemes may be established without local assumptions.

It is important to note that the fractional Gronwall-type inequality mentioned above cannot be directly applied to time-fractional differential equations with time delay. Recently, Li et al. [9] developed a new fractional Gronwall-type inequality for fractional problems, in order to analyze fractional reaction–diffusion equations with fixed temporal delay. This novel form of Gronwall's inequality helps in obtaining an optimal error estimate for fixed time-delay fractional parabolic equations. The task is achieved employing arguments from other works [10–13] which employ local assumptions in time. The present work aims at extending the results in [7, 9] to study time-fractional differential equations with functional delay. Such systems can effectively model physical problems for which the evolution does not only depend on the present state of the system but also on the past history. Those equations provide more realistic models for phenomena that display time-lags or memory-effects. As examples of those phenomena in the sciences, we may cite various problems in automatic control [14], traffic models [15] and population dynamics [16].

Throughout this manuscript, we will consider one-dimensional time-fractional parabolic differential equation with functional delay of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + f(x, t, u(x, t), u_t(x, \cdot)). \tag{1}$$

Here, $0 < \alpha \leq 1$ and the fractional derivative is understood in the sense of Caputo. Moreover, $x \in [0, X] \subset \mathbb{R}$ and $t \in [t_0, \theta] \subset \mathbb{R}$ are independent variables in space and time, respectively. Also, $u(x,t)$ is the solution of the problem, $u_t(x,\cdot) = \{u(x,t+s): -\tau \leq s < 0\}$ is the prehistory of the solution by the time t, and τ is the value of the temporal delay. Furthermore, we will impose initial-boundary conditions of the form

$$\begin{cases} u(x, t) = \varphi(x, t), & (x, t) \in [0, X] \times [t_0 - \tau, t_0], \\ u(0, t) = u(X, t) = 0, & t \in [t_0, \theta]. \end{cases} \tag{2}$$

This problem is a temporal fractional-order generalization of that proposed in [17]. In that work, one-dimensional parabolic equations with temporal delay effects were considered for the case of variable coefficients of heat conductivity. For more works on difference schemes for equations with functional delay, we refer to [18, 19].

Gronwall-type inequalities are crucial in the qualitative analysis of fractional systems in differential and difference settings [20]. In the differential setting, the existence and uniqueness of positive solutions for a class of nonlinear fractional delay differential equations was considered in [21] using a nonlinear alternative of the Leray–Schauder type. The stability and the dissipation of Caputo nonlinear fractional functional differential equations with order $0 < \alpha < 1$ were discussed in [22], based on new Gronwall-type inequalities. Also, the dissipation for the time-fractional nonlinear sub-diffusion equation was studied in [23], using generalized Gronwall inequalities. Moreover, some attractiveness results for fractional functional differential equations were obtained in [24] using a fixed-point theorem. Finally, a robust finite-time stability problem of fractional-order systems with time-varying delay and nonlinear perturbation was investigated in [25] based on a generalized Gronwall inequality. A delay-dependent sufficient condition for robust finite-time stability of such systems was provided there in terms of the Mittag–Leffler function.

On the other hand, optimal error estimates of some numerical schemes for multidimensional nonlinear time-fractional Schrödinger equation have been obtained using a discrete form of a fractional Gronwall-type inequality introduced in [7], and numerical solutions were proposed in [9] for nonlinear time-fractional reaction–diffusion equations with fixed-time delay in terms of a linearized compact finite-difference scheme. The convergence and the stability of the proposed scheme were obtained in terms of a new fractional Gronwall-type inequality. In [26], the authors considered a technique of creation of difference schemes for time- and space-fractional partial differential equations with delay effect on time. The algorithm is a fractional analogue of the pure implicit numerical method in which the model is reduced on each time-step to the solution of linear algebraic systems. In the present work, we extend the role of the discrete form of the fractional Gronwall-type inequality proposed firstly in [7], in order to obtaining optimal error estimates. Moreover, we will study the convergence and stability of the difference method using those results.

1.1 | Main assumptions

Throughout this work, we assume that $\varphi(x,t)$ and the functional f are chosen such (1) and (2) has a unique solution $u(x,t)$ in the classical sense. Moreover, we assume that the function $u(x,t)$ is sufficiently smooth in order to satisfy the requirements for consistency and convergence. We denote by $Q = Q([-\tau,0))$ the set of functions $u(s)$ that are piecewise continuous on the interval $[-\tau,0)$, with a finite number of points of discontinuity of the first kind and right continuous at the points of discontinuity. In addition, the functions $u(s)$ have a finite left-hand limit at zero. We define the norm of a function $u \in Q$ by the identity

$$\|u(\cdot)\|_Q = \sup_{-\tau\leq s<0} \mid u(s) \mid . \tag{3}$$

We additionally suppose that the functional $f(x,t,u,v)$ is real-valued and defined on $[0,X] \times [t_0,\theta] \times R \times Q$. Moreover, we suppose that f is Lipschitz with respect to the last two arguments, that is, there exists a constant L_f such that, for all $x \in [0,X]$, $t \in [t_0,\theta]$, $u^1,u^2 \in R$ and $v^1[\cdot],v^2[\cdot] \in Q$, the following inequality holds:

$$\mid f(x,t,u^1,v^1(.)) - f(x,t,u^2,v^2(.)) \mid\leq L_f(|u^1 - u^2| + \|v^1(.) - v^2(.)\|_Q). \tag{4}$$

With this assumptions, a discretization of (1) and (2) will be proposed in Section 2. Interpolation and extrapolation constructions will be provided therein, in order to describe the approximation of the nonlinear term with functional delay. Also, the concept of the residual of a method with interpolation and its associated order will be discussed. Section 3 reports on the main theoretical results of this work, namely, the existence and uniqueness of solutions, estimates of the order of convergence of the scheme and the global stability property. Some numerical simulations will be provided in Section 4, in order to confirm the convergence properties of the numerical model. Finally, this manuscript closes with a section of concluding remarks.

2 | NUMERICAL MODEL

The time-discretization of the problem (1) and (2) will be performed here using a combination of an L_1 approximation to the Caputo fractional derivative and, at the same time, an interpolation–extrapolation operator of the discrete prehistory. A compact difference operator for the spatial discretization is employed. To that end, let us divide the spatial interval $[0,X]$ into equally spaced nodes with step $h = X/N$, namely, $x_i = ih$, for each $i \in \{0,1, \dots, N\}$. Meanwhile, the temporal interval $[t_0,\theta]$ will be partitioned into subintervals of length $\Delta > 0$, with the respective nodes given by $t_k = t_0 + k\Delta$, for each $k \in \{0,1, \dots, M\}$. Without lose of generality, we will assume that the value $\tau/\Delta = m$ is an integer. Use U_k^i to denote an approximation to the value of the function u at the node (x_i,t_k).

Definition 1 Let $i \in \{0,1, \ldots, N\}$ and $k \in \{1, \ldots, M\}$. The *L_1 approximation formula* for the Caputo fractional derivative of order $0 < \alpha \leq 1$ at the node (x_i,t_k) is defined as

$$D_\Delta^L U_k^i = \frac{\Delta^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} a_{k-j}(U_j^i - U_{j-1}^i), \quad \text{where } a_i = (i+1)^{1-\alpha} - i^{1-\alpha}. \tag{5}$$

In the sequel, we will consider the following linear operator, defined for each $k \in \{0,1, \ldots, M\}$:

$$\mathscr{A}_h U_k^i = \begin{cases} \left(1 + \frac{h^2}{12}\delta_x^2\right) U_k^i, & 1 \leq i \leq N-1, \\ U_k^i, & i \in \{0, N\}. \end{cases} \tag{6}$$

Lemma 1 (Liao and Sun [27]). *If $g(x) \in C^6[0,X]$ and $\zeta(\lambda) = 5(1-\lambda)^3 - 3\lambda^5$ then*

$$\mathscr{A}_h g''(x_i) = \delta_x^2 g(x_i) + \frac{h^4}{360} \int_0^1 (g^{(6)}(x_i - \lambda h) + g^{(6)}(x_i + \lambda h))\zeta(\lambda) d\lambda, \quad \forall i = 1, \ldots, N-1, \tag{7}$$

where $\delta_x^2 g^i = \frac{1}{h^2}(g^{i+1} - 2g^i + g^{i-1})$.

Applying the operator (6) on both sides of (1) and at the grid point (x_i,t_k), we readily obtain

$$\mathscr{A}_h \frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) = \mathscr{A}_h \frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \mathscr{A}_h f(x_i, t_k, u(x_i, t_k), u_{t_k}(x_i, \cdot)). \tag{8}$$

Employing now the notation U_k^i instead of $u(x_i,t_k)$, using the L_1 approximation scheme (5) along with the definition of the linear operator $\mathscr{A}_h$ and the property summarized in Lemma 1, we reach

$$\mathscr{A}_h D_\Delta^L U_k^i = \delta_x^2 U_k^i + \mathscr{A}_h f(x_i, t_k, U_k^i, U_{t_k}^i(\cdot)) + R_k^i. \tag{9}$$

Here, the truncation error satisfies

$$R_k^i = \frac{h^4}{360} \int_0^1 (u^{(6)}(x_i - \lambda h) + u^{(6)}(x_i + \lambda h))\zeta(\lambda) d\lambda + \mathscr{A}_h \left(D_\Delta^L U_k^i - \frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) \right). \tag{10}$$

In order to approximate the prehistory function $u_t(x,\cdot) = \{u(x,t+s)\colon -\tau \leq s < 0\}$, we introduce a discrete prehistory at the time t_k, for each $k \in \{0,1, \ldots, M\}$. More precisely, we consider $\{u_l^i\}_k = \{u_l^i : k-m \leq l \leq k\}$ for each $i \in \{0,1, \ldots, N\}$. Next, an interpolation–extrapolation operator $\mathbf{I}$ defined on the discrete prehistory will map each time t_k and each discrete prehistory $\{u_l^i\}_k$ to a function $v_k^i(\cdot) \in Q([-\tau, \Delta])$, for each $k \in \{0,1, \ldots, M\}$. In symbols,

$$\mathbf{I} : \{u_l^i\}_k \underset{(t_k : k \in \{0,1,\ldots,M\})}{\rightarrow} v_k^i(\cdot) \in Q([-\tau, \Delta]). \tag{11}$$

In what follows, we will omit the index k at the function $v_k^i(\cdot)$. We define next the order of $\mathbf{I}$.

Definition 2 An interpolation–extrapolation operator has *error order* Δ^p with respect to the exact solution if there exist constants C_1 and C_2 such that, for all i,k and $t \in [t_k - \tau, t_{k+1}]$, the following inequality is satisfied:

$$\|v^i(t) - u(x_i, t)\| \leq C_1 \max_{k-m \leq l \leq k} |u_l^i - u(x_i, t_l)| + C_2 \Delta^p. \tag{12}$$

In order to satisfy the convergence requirements of the method, we employ here the second-order piecewise linear interpolation scheme (see [17])

$$v^i(t_k + s) = ((t_l - t_k - s)u_{l-1}^i + (t_k + s - t_{l-1})u_l^i)/\Delta, \quad \text{if} t_{l-1} \leq t_k + s \leq t_l, \tag{13}$$

together with the extrapolation by extension

$$v^i(t_k+s) = ((-s)u_{k-1}^i + (\Delta+s)u_k^i)/\Delta, \quad \text{if } t_k \le t_k+s \le t_{k+1}. \tag{14}$$

For simplicity, we use $F(v(\cdot))$ to approximate the functional $f(x,t,u(x,t),u_t(x,\cdot))$ at the grid nodes. Denote by $F_k^i(v^i(\cdot))$ the value of $F(v(\cdot))$ at the node (x_i,t_k), so that $F_k^i(v^i(\cdot)) = f(x_i,t_k,v^i(t_k),v^i_{t_{k-1}+\Delta}(\cdot))$. In this case, the functional $F_k^i(v^i(\cdot))$ is defined on $Q([-\tau,\Delta])$, and is Lipschitz with respect to $v(\cdot)$, with constant L_f.

As a consequence of this approach, the approximation of the nonlinear source term gives

$$\mathscr{A}_h D_\Delta^L U_k^i = \delta_x^2 U_k^i + \mathscr{A}_h F_k^i(v^i(\cdot)) + \Psi_k^i. \tag{15}$$

Omitting the (small term) error of approximation Ψ_k^i arising after interpolation, and replacing then the function U_k^i by its numerical approximation u_k^i, we reach the following final form of the difference scheme:

$$\mathscr{A}_h D_\Delta^L u_k^i = \delta_x^2 u_k^i + \mathscr{A}_h F_k^i(v^i(\cdot)), \quad \forall i \in \{1,\dots,N-1\}, \forall k \in \{0,1,\dots,M-1\}. \tag{16}$$

Moreover, we consider initial-boundary approximations of the form

$$\begin{cases} u_0^i = \phi(x_i,t_0), & \forall i \in \{0,1,\dots,N\}, \\ u_k^0 = u_k^N = 0, & \forall k \in \{0,1,\dots,M\}. \end{cases} \tag{17}$$

Note that the value of the functional $F_k^i(v^i(\cdot))$ is calculated explicitly by means of interpolation and extrapolation, with

$$v^i(\cdot) = \begin{cases} \frac{1}{\Delta}((t_l - t_{k-1} - (\Delta+s))u_{l-1}^i + (t_{k-1}+(\Delta+s)-t_{l-1})u_l^i)), & t_{l-1} \le t_{k-1}+(\Delta+s) \le t_l, 1 \le l \le k-1, \\ \frac{1}{\Delta}((-(\Delta+s))u_{k-2}^i + (\Delta+(\Delta+s))u_{k-1}^i), & t_{k-1} \le t_{k-1}+(\Delta+s) \le t_k, \\ \phi(x_i,t), & 2t_0 - t_m \le t_{k-1}+(\Delta+s) \le t_0. \end{cases} \tag{18}$$

We turn our attention now to the study of the approximation (residual) error of the difference method (16). For each $i \in \{0,1,\dots,N\}$, introduce the discrete prehistory of the exact solution at t_j as $\{u(x_i,t_k)\}_j = \{u(x_i,t_k)\colon j-m \le k \le j\}$, for each $j \in \{0,1,\dots,M\}$. Moreover, we will employ piecewise-linear interpolation with extrapolation by continuation of the true form $u_{tk}(x_i,\cdot)$. More precisely,

$$u_{t_k}(x_i,\cdot) = \begin{cases} \frac{1}{\Delta}((t_l - t_{k-1} - (\Delta+s))u(x_i,t_{l-1}) & t_{l-1} \le t_{k-1}+(\Delta+s) \le t_l \\ \quad +(t_{k-1}+(\Delta+s)-t_{l-1})u(x_i,t_l)), & \text{and} \quad 1 \le l \le k-1, \\ \frac{1}{\Delta}((-(\Delta+s))u(x_i,t_{k-2}) + (\Delta+(\Delta+s))u(x_i,t_{k-1})), & t_{k-1} \le t_{k-1}+(\Delta+s) \le t_k, \\ \phi(x_i,t), & 2t_0 - t_m \le t_{k-1}+(\Delta+s) \le t. \end{cases} \tag{19}$$

Definition 3 The *residual (with interpolation)* of the difference method (16) is defined as the value

$$\Psi_k^i = \mathscr{A}_h D_\Delta^L u(x_i,t_k) - \delta_x^2 u(x_i,t_k) - \mathscr{A}_h F_k^i(u_{t_k}(x_i,\cdot)). \tag{20}$$

It is possible to determine the order of the residual of the method for specific functionals F by using the Taylor expansion of the function $u(x,t)$ under suitable smoothness conditions. The following result is an analog of that corresponding to difference schemes without delay [28] and by the aid of

Lemma 1 under the assumption that L_1-type approximations of the Caputo fractional derivative of the solution is smooth when the initial time $t_0 = 0$.

Theorem 1 (Lekomtsev and Pimenov [17]). *Let the exact solution satisfy the condition* $u(x, t) \in C^{(6,2)}([0, X] \times [t_0, \theta])$ *and the partial derivatives* $f_u(x,t,u,v)$ *and* $f_v(x,t,u,v)$ *are continuous in the* ϵ_0*-neighborhood of the solution, where* ϵ_0 *is a positive constant. If* $f(x_i, t_k, v^i(t_k), v^i_{t_k}(\cdot))$ *is used to approximate* $f(x,t,u(x,t),u_t(x,\cdot))$ *at the grid nodes then the residual of the method* (16) *and* (17) *has order* $\Delta^{2-\alpha} + h^4$.

3 | EFFICIENCY ANALYSIS

In this section, we will establish the most important numerical properties of the numerical model (16) and (17). More precisely, we prove the unique solubility of the difference method along with the convergence and stability.

Theorem 2 (Existence and uniqueness). *The difference method* (16) *and* (17) *is uniquely solvable*.

Proof. Notice that the numerical model can be written in a matrix form as

$$D^L_\Delta u_k = \widehat{\mathbf{M}} u_k + F^i_k(v^i(\cdot)), \quad 1 \le k \le M. \tag{21}$$

Here, $\widehat{\mathbf{M}} = \mathbf{M}^{-1}\mathbf{B}$, and

$$\mathbf{M} = \frac{1}{12}\begin{bmatrix} 10 & 1 & 0 & \dots & 0 & 0 \\ 1 & 10 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 10 \end{bmatrix}, \quad \mathbf{B} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}. \tag{22}$$

It is easily shown that the coefficient matrix $\widehat{\mathbf{M}}$ is symmetric and positive definite at each time level t_k. As a consequence, the existence and the uniqueness of the solutions readily follow. ∎

For the remainder of this manuscript, we let $\mathscr{V}_h = \{v : v = (v_0, v_1, \dots, v_N), v_0 = v_N = 0\}$. Moreover, if $uv \in \mathscr{V}_h$ then we will use the notation $u \le v$ to represent that $u_i \le v_i$, for each $i \in \{1, \dots, N-1\}$.

Definition 4 For any $u, v \in V_h$, we define

$$(u, v) = h \sum_{i=1}^{N-1} u_i v_i, \tag{23}$$

$$|u|_1^2 = h \sum_{i=1}^{N} \left(\delta_x u_{i-\frac{1}{2}}\right)^2, \tag{24}$$

$$\langle v, w \rangle_{\mathscr{A}} = h \sum_{i=1}^{N} \delta_x v^{i-\frac{1}{2}} \delta_x w^{i-\frac{1}{2}} - \frac{h^2}{12} h \sum_{i=1}^{N-1} \delta_x^2 v^i \delta_x^2 w^i. \tag{25}$$

Moreover, we let $\|u\|^2 = (u, u)$, $\|u\|_\infty = \max_{0 \le i \le N} | u_i|$ and $\|v\|^2_{\mathscr{A}} = \langle v, v\rangle_{\mathscr{A}}$.

Lemma 2 (Samarskii and Andreev [28]). *If* $u \in \mathscr{V}_h$ *then* $\|u\|_\infty \le \frac{\sqrt{X}}{2}|u|_1$ *and* $\|u\| \le \frac{X}{\sqrt{6}}|u|_1$.

Lemma 3 (Gao and Sun [29]). *If* $u \in \mathscr{V}_h$ *then* $\frac{2}{3}|u|_1^2 \le \langle u, u\rangle_{\mathscr{A}} \le |u|_1^2$.

Lemma 4 (Gao and Sun [29]). *If* $v, w \in \mathscr{V}_h$ *then* $\langle v, w\rangle_{\mathscr{A}} = -h\sum_{i=1}^{N-1}(\mathscr{A}_h v^i)\delta_x^2 w^i$.

Lemma 5 (Li et al. [8]). *Let* $\{p_n\}_{n=0}^{\infty}$ *be a sequence defined as* $p_0 = 1$, *and* $p_n = \sum_{j=1}^{n}(a_{j-1} - a_j)p_{n-j}$ *for each* $\forall n \ge 1$.

a *Let* $0 < p_n < 1$ *for each* $n \ge 0$. *Then* $\sum_{j=k}^{n} p_{n-j}a_{j-k} = 1$ *holds for each* $n \ge 1$ *and* $1 \le k \le n$.
b *If* $n \ge 1$ *then*

$$\Gamma(2-\alpha)\sum_{j=1}^{n} p_{n-j} \le \frac{n^\alpha}{\Gamma(1+\alpha)}, \tag{26}$$

$$\frac{\Gamma(2-\alpha)}{\Gamma(1+(k-1)\alpha)}\sum_{j=1}^{n-1} p_{n-j}j^{(k-1)\alpha} \le \frac{n^{k\alpha}}{\Gamma(1+k\alpha)}, \quad \forall k \in \mathbb{N}. \tag{27}$$

Lemma 6 (Li et al. [8]). *Let* $\vec{e} = [1, 1, \dots, 1]^T \in \mathbb{R}^n$, *and let*

$$J = 2\Gamma(2-\alpha)\lambda\Delta^\alpha \begin{bmatrix} 0 & p_1 & \dots & p_{n-2} & p_{n-1} \\ 0 & 0 & \dots & p_{n-3} & p_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & p_1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{n\times n}. \tag{28}$$

Then $J^i = 0$ *for each* $i \ge n$. *Moreover, the following relations are satisfied*:

$$J^k\vec{e} \le \vec{B}, \quad \vec{B} = \frac{1}{\Gamma(1+k\alpha)}[(2\lambda t_n^\alpha)^k, (2\lambda t_{n-1}^\alpha)^k, \dots, (2\lambda t_1^\alpha)^k]^\top, \quad \forall k \in \mathbb{N}\cup\{0\}, \tag{29}$$

$$\sum_{j=0}^{i} J^i\vec{e} = \sum_{j=0}^{n-1} J^j\vec{e} \le \vec{B}_E, \quad \vec{B}_E = [E_\alpha(2\lambda t_n^\alpha), E_\alpha(2\lambda t_{n-1}^\alpha), \dots, E_\alpha(2\lambda t_1^\alpha)]^\top. \tag{30}$$

In the following, we will set $\varepsilon_k^i = u(x_i, t_k) - U_k^i$ for each $i \in \{0, 1, \dots, N\}$ and $k \in \{0, 1, \dots, M\}$. Notice that the residual with interpolation of the difference method (16) can be rewritten as

$$\mathscr{A}_h D_\Delta^L u(x_i, t_k) = \delta_x^2 u(x_i, t_k) + \mathscr{A}_h F_k^i(u_{t_k}(x_i, \cdot)) + \Psi_k^i. \tag{31}$$

Subtracting (16) from (31), we obtain the following expression for the error of the difference model:

$$\mathscr{A}_h D_\Delta^L \varepsilon_k^i = \delta_x^2 \varepsilon_k^i + \mathscr{A}_h(F_k^i(u_{t_k}(x_i, \cdot)) - F_k^i(v^i(\cdot))) + \Psi_k^i. \tag{32}$$

Definition 5 The difference model (16) and (17) converges with order $h^{q_1} + \Delta^{q_2}$ if there exists a constant C independent of ε_k^i, h and Δ such that $|\varepsilon_k^i| \le C(h^{q_1} + \Delta^{q_2})$ for all $i \in \{0, 1, \dots, N\}$ and $k \in \{0, 1, \dots, M\}$.

Theorem 3 (Convergence). *Let $(u_k)_{k=0}^{M}$ be the unique solution of the algebraic system* (16) *and* (17). *Then there exists a positive constant C satisfying $\|\varepsilon_k\|_\infty \le C(\Delta^{2-\alpha} + h^4)$.*

Proof. Multiplying each side of (32) by $-h\delta_x^2\varepsilon_k^i$ and adding over all indexes i from 1 to $N-1$ yield

$$-\sum_{i=1}^{N-1}(h\delta_x^2\varepsilon_k^i)\mathscr{A}_h(D_\Delta^L\varepsilon_k^i) = -h\sum_{i=1}^{N-1}(\delta_x^2\varepsilon_k^i)^2 - \sum_{i=1}^{N-1}(h\delta_x^2\varepsilon_k^i)\Psi_k^i - \sum_{i=1}^{N-1}(h\delta_x^2\varepsilon_k^i)\mathscr{A}_h(F_k^i(u_{t_k}(x_i,\cdot)) - F_k^i(v^i(\cdot))). \tag{33}$$

Using summation by parts and Lemma 4 on the left-hand side of (33) followed by an application of (17), we obtain

$$\begin{aligned}
-\sum_{i=1}^{N-1}(h\delta_x^2\varepsilon_k^i)\mathscr{A}_h(D_\Delta^L\varepsilon_k^i) &= h\sum_{i=1}^{N-1}(-\delta_x^2\varepsilon_k^i)\left(1+\frac{h^2}{12}\delta_x^2\right)\left(\varepsilon_k^i - \sum_{j=1}^{k-1}(a_{k-j-1}-a_{k-j})\varepsilon_j^i - a_{k-1}\varepsilon_0^i\right)\\
&= \frac{h}{\Delta^\alpha\Gamma(2-\alpha)}\sum_{i=0}^{N-1}\left(\delta_x\varepsilon_k^{i+\frac{1}{2}}\right)\left(\delta_x\varepsilon_k^{i+\frac{1}{2}} - \sum_{j=1}^{k-1}(a_{k-j-1}-a_{k-j})\delta_x\varepsilon_j^{i+\frac{1}{2}} - a_{k-1}\delta_x\varepsilon_0^{i+\frac{1}{2}}\right)\\
&\quad - \frac{h^2}{12\Delta^\alpha\Gamma(2-\alpha)}h\sum_{i=0}^{N-1}(\delta_x^2\varepsilon_k^i)\left(\delta_x^2\varepsilon_k^i - \sum_{j=1}^{k-1}(a_{k-j-1}-a_{k-j})\delta_x^2\varepsilon_j^i - a_{k-1}\delta_x^2\varepsilon_0^i\right)\\
&= \frac{1}{\Delta^\alpha\Gamma(2-\alpha)}\left(\langle\varepsilon_k,\varepsilon_k\rangle_{\mathscr{A}} - \sum_{j=1}^{k-1}(a_{k-j-1}-a_{k-j})\langle\varepsilon_k,\varepsilon_j\rangle_{\mathscr{A}} - a_{k-1}\langle\varepsilon_k,\varepsilon_0\rangle_{\mathscr{A}}\right)\\
&\ge \frac{1}{\Delta^\alpha\Gamma(2-\alpha)}\left(\|\varepsilon_k\|_{\mathscr{A}}^2 - \sum_{j=1}^{k-1}(a_{k-j-1}-a_{k-j})\frac{\|\varepsilon_k\|_{\mathscr{A}}^2+\|\varepsilon_j\|_{\mathscr{A}}^2}{2} - a_{k-1}\frac{\|\varepsilon_k\|_{\mathscr{A}}^2+\|\varepsilon_0\|_{\mathscr{A}}^2}{2}\right)\\
&= \frac{1}{2}D_\Delta^L\|\varepsilon_k\|_{\mathscr{A}}^2.
\end{aligned} \tag{34}$$

For any $k \in \{0, 1, \ldots, M\}$, the layer-by-layer error is defined by $\varepsilon_k = (\varepsilon_k^1, \varepsilon_k^2, \ldots, \varepsilon_k^{N-1})$ with $\|\varepsilon_k\| = \max_{1\le i\le N-1}|\varepsilon_k^i|$. The prehistory of a layer-by-layer error at the time t_k is given as $\{\varepsilon_l\}_k = \{\varepsilon_l : 0 \le l \le k\}$, for each $k \in \{0, 1, \ldots, M\}$. We also let $\|\{\varepsilon_l\}\|_k = \max_{0\le l\le k}\|\varepsilon_l\|$. Apply the Cauchy–Schwarz inequality to the last two terms at the right-hand side of (33) and use (4) depending on the previous definitions of a layer-by-layer error to obtain

$$\begin{aligned}
&-\sum_{i=1}^{N-1}(h\delta_x^2\varepsilon_k^i)\Psi_k^i - \sum_{i=1}^{N-1}(h\delta_x^2\varepsilon_k^i)\mathscr{A}_h(F_k^i(u_{t_k}(x_i,\cdot)) - F_k^i(v^i(\cdot)))\\
&\le h\sum_{i=1}^{N-1}(\delta_x^2\varepsilon_k^i)^2 + \frac{1}{2}\sum_{i=1}^{N-1}h(L_f|\{\varepsilon_l\}_k|)^2 + \frac{1}{2}\sum_{i=1}^{N-1}h|\Psi_k^i|^2.
\end{aligned} \tag{35}$$

Substitute now (34) and (35) into (33), and apply then Theorem 1. Apply next Lemmas 2 and 3 to note that

$$D_\Delta^L\|\varepsilon_k\|_{\mathscr{A}}^2 \le L_f^2\|\{\varepsilon_l\}_k\|_{\mathscr{A}}^2 + C_1(\Delta^{2-\alpha}+h^4)^2 \le \frac{L_f^2X^2}{4}\|\{\varepsilon_l\}_k\|_{\mathscr{A}}^2 + C_1(\Delta^{2-\alpha}+h^4)^2. \tag{36}$$

Here, C_1 is a positive constant which is independent of h and Δ. Substitute now (5) into (36), multiply then both ends of the resulting inequality by the constant $p_{n\text{-}k}$ defined in

Lemma 5 and sum finally over all indexed k from 1 to n to reach the inequality

$$\sum_{k=1}^{n} p_{n-k} \sum_{j=1}^{k} a_{k-j}\delta_t \|\varepsilon_j\|_{\mathscr{A}}^2 \leq \Delta^\alpha \Gamma(2-\alpha) \frac{(L_f X)^2}{4} \sum_{k=1}^{n} p_{n-k} \|\{\varepsilon_l\}_k\|_{\mathscr{A}}^2 + \Delta^\alpha \Gamma(2-\alpha) C_1 (\Delta^{2-\alpha} + h^4)^2 \sum_{k=1}^{n} p_{n-k}. \tag{37}$$

On the other hand, Lemma 5 yields

$$\sum_{k=1}^{n} p_{n-k} \sum_{j=1}^{k} a_{k-j}\delta_t \|\varepsilon_j\|_{\mathscr{A}}^2 = \sum_{j=1}^{n} \delta_t \|\varepsilon_j\|_{\mathscr{A}}^2 \sum_{k=j}^{n} p_{n-k} a_{k-j} = \|\varepsilon_n\|_{\mathscr{A}}^2 - \|\varepsilon_0\|_{\mathscr{A}}^2, \quad n \geq 1, \tag{38}$$

$$\Delta^\alpha \Gamma(2-\alpha) C_1 (\Delta^{2-\alpha} + h^4)^2 \sum_{k=1}^{n} p_{n-k} \leq \frac{t_n^\alpha}{\Gamma(1+\alpha)} C_1 (\Delta^{2-\alpha} + h^4)^2. \tag{39}$$

Substitute (38) and (39) into (37), and convey that

$$\|\varepsilon_0\|_{\mathscr{A}}^2 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} C_1 (\Delta^{2-\alpha} + h^4)^2 = \zeta_n. \tag{40}$$

Notice that $\zeta_n \geq \zeta_k$ is satisfied whenever $n \geq k \geq 1$. Under these circumstances, if we let $\Delta^\alpha \leq [2\Gamma(2-\alpha)\lambda]^{-1}$ and $\lambda = \frac{1}{4}(L_f X)^2$ then the following estimate is valid in general:

$$\|\{\varepsilon_l\}_r\|_{\mathscr{A}}^2 \leq 2\zeta_n + 2\Delta^\alpha \Gamma(2-\alpha)\lambda \sum_{k=1}^{n-1} p_{n-k} \|\{\varepsilon_l\}_k\|_{\mathscr{A}}^2, \quad r = 1, 2, \ldots, n. \tag{41}$$

Let $\mathbf{F} = [\|\{\varepsilon_l\}_n\|_{\mathscr{A}}^2, \|\{\varepsilon_l\}_{n-1}\|_{\mathscr{A}}^2, \ldots, \|\{\varepsilon_l\}_1\|_{\mathscr{A}}^2]^{\top}$. Then (41) can be written in vector form as

$$\mathbf{F} \leq 2\zeta_n \vec{e} + J\mathbf{F} \leq J(J\mathbf{F} + 2\zeta_n \vec{e}) + 2\zeta_n \vec{e} = J^2 \mathbf{F} + 2\zeta_n \sum_{j=0}^{1} J^j \vec{e} \leq \cdots \leq J^n \mathbf{F} + 2\zeta_n \sum_{j=0}^{n-1} J^j \vec{e}, \tag{42}$$

where $\vec{e}$ and J are defined in Lemma 6. Using the properties of Lemma 6, we deduce from (42) that

$$\|\{\varepsilon_l\}_n\|_{\mathscr{A}}^2 \leq 2\left(\|\varepsilon_0\|_{\mathscr{A}}^2 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} C_1 (\Delta^{2-\alpha} + h^4)^2 \right) E_\alpha(2\lambda t_n^\alpha), \tag{43}$$

where $E_\alpha(z)$ is the Mittag–Leffler function. The conclusion of the theorem readily follows now. ■

Next, we analyze the numerical stability of the compact difference scheme (16) and (17). The numerical stability means that the initial value has a small perturbation implies the numerical solution has a small perturbation. For this purpose, we suppose that v_k^i be the solution of

$$\mathscr{A}_h D_\Delta^L v_k^i = \delta_x^2 v_k^i + \mathscr{A}_h F_k^i(\omega^i(\cdot)), \quad \forall i \in \{1, \ldots, N-1\}, \forall k \in \{0, 1, \ldots, M-1\}, \tag{44}$$

$$\begin{cases} v_0^i = \phi(x_i, t_0) + \rho_k^i, & \forall i \in \{0, 1, \ldots, N\}, \\ v_k^0 = v_k^N = 0, & \forall k \in \{0, 1, \ldots, M\}. \end{cases} \tag{45}$$

where ρ_k^i is a small perturbation of $\phi(x,t)$.

Definition 6 A numerical scheme (16) and (17) is stable if the discrete numerical solutions u_k^i satisfying (16) and (17) and v_k^i satisfying (44) and (45) are such that

$$\|\eta_k\|_\infty = \|u_k^i - v_k^i\|_\infty \leq \overline{C} \max_{2t_0 - t_m \leq j \leq t_0} |\rho_j|_1$$

where $\overline{C}$ is a bounded constant which is independent of h and Δ.

Accordingly, we have the following stability theorem.

Theorem 4 (Stability). *The difference scheme* (16) *and* (17) *is stable with respect to the initial perturbation of* $\phi(x_i,t_0)$, *such that*

$$\|\eta_k\|_\infty \leq \widehat{C} \max_{2t_0-t_m\leq j\leq t_0} |\rho_j|_1,$$

where $\widehat{C}$ *is a bounded constant which is independent of h and* Δ.

Proof. . Subtracting (44) and (45) form (16) and (17), then we have

$$\mathscr{A}_h D^L_\Delta \eta^i_k = \delta^2_x \eta^i_k + \mathscr{A}_h(F^i_k(v^i(\cdot)) - F^i_k(\omega^i(\cdot))), \tag{46}$$

$$\begin{cases} \eta^i_0 = \rho^i_k, & \forall i \in \{0,1,\dots,N\}, \\ \eta^0_k = \eta^N_k = 0, & \forall k \in \{0,1,\dots,M\}. \end{cases} \tag{47}$$

Multiplying each side of (32) by $-h\delta^2_x\eta^i_k$ and adding over all indexes i from 1 to $N-1$ and continue step by step as in the proof of Theorem 3 to complete the proof. ∎

4 | NUMERICAL SIMULATIONS

The purpose of the present section is to verify the convergence rate of the method. The simulations of this section were obtained using an implementation of our method in ©Matlab 8.5.0.197613 (R2015a) on a ©Hewlett-Packard 6005 Pro Microtower desktop computer with Linux Mint 18 “Sylvia” Cinnamon edition. We will consider the absolute error at the time T between the exact solution u of the continuous problem and the corresponding approximations U, which is given by

$$\varepsilon_{\Delta,h} = |\ \|u - U|\|_\infty. \tag{48}$$

Moreover, we will consider the standard rates

$$\rho^x_{\Delta,h} = \log_2\left(\frac{\varepsilon_{\Delta,2h}}{\varepsilon_{\Delta,h}}\right), \quad \rho^t_{\Delta,h} = \log_2\left(\frac{\varepsilon_{2\Delta,h}}{\varepsilon_{\Delta,h}}\right). \tag{49}$$

Example 1. Fix $X = \theta = 1$, $t_0 = 0$ and $\tau = 0.1$, and consider the problem (1) and (2) with

$$f(x,t,u(x,t),u_t(x,\cdot)) = -2u(x,t) + \frac{u(x,t-0.1)}{1+u^2(x,t-0.1)} + g(x,t), \quad \forall (x,t) \in [0,1]\times[0,1]. \tag{50}$$

Here, g is a function such that the solution of the continuous problem is $u(x,t) = t^2\sin(\pi x)$. As initial data, we choose $\phi(x,t) = t^2\sin(\pi x)$, for each $(x,t) \in [0,1]\times[-0.1,0]$. It is easy to check that the function g is given by

$$g(x,t) = \left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} + \pi^2t^2 + 2t^2 - \frac{(t-0.1)^2}{1+(t-0.1)^2\sin(\pi x)}\right]\sin(\pi x), \quad \forall (x,t) \in [0,1]\times[0,1]. \tag{51}$$

Using this model, the upper half of Table 1 shows the spatial analysis of convergence of the numerical solution of the continuous problem, for various values of the computational parameters and $\alpha = 0.5$. The results confirm the quartic spatial order of

TABLE 1 Table of absolute errors and standard convergence rates in space when approximating the solution u of (1) and (2) with $\alpha = 0.5$, using the difference method (16) and (17)

Spatial analysis of convergence

		$t = 0.5$		**$t = 1$**	
Δ	**h**	$\epsilon_{\Delta,h}$	$\rho^x_{\Delta,h}$	$\epsilon_{\Delta,h}$	$\rho^x_{\Delta,h}$
0.04	0.02×2^{-1}	3.2751×10^{-4}	—	3.6026×10^{-4}	—
	0.02×2^{-2}	3.5989×10^{-5}	3.1859	3.9849×10^{-5}	3.1764
	0.02×2^{-3}	3.2546×10^{-6}	3.4670	3.5526×10^{-6}	3.4876
	0.02×2^{-4}	2.4424×10^{-7}	3.7361	2.7513×10^{-7}	3.6907
0.02	0.02×2^{-1}	1.4225×10^{-4}	—	1.6069×10^{-4}	—
	0.02×2^{-2}	1.2723×10^{-5}	3.4829	1.6649×10^{-5}	3.2708
	0.02×2^{-3}	9.5078×10^{-7}	3.7422	1.3757×10^{-6}	3.5972
	0.02×2^{-4}	6.3720×10^{-8}	3.8993	9.8052×10^{-8}	3.8105
0.01	0.02×2^{-1}	5.5103×10^{-5}	—	6.6373×10^{-5}	—
	0.02×2^{-2}	4.2365×10^{-6}	3.7012	5.4627×10^{-6}	3.6029
	0.02×2^{-3}	2.8970×10^{-7}	3.8702	3.7694×10^{-7}	3.8572
	0.02×2^{-4}	1.8711×10^{-8}	3.9526	2.4809×10^{-8}	3.9254

Temporal analysis of convergence

		$t = 0.5$		**$t = 1$**	
h	**Δ**	$\epsilon_{\Delta,h}$	$\rho^t_{\Delta,h}$	$\epsilon_{\Delta,h}$	$\rho^t_{\Delta,h}$
0.0100	0.01×2^{-1}	2.1879×10^{-5}	—	2.4731×10^{-5}	—
	0.01×2^{-2}	8.4635×10^{-6}	1.3702	9.8871×10^{-6}	1.3227
	0.01×2^{-3}	3.2462×10^{-6}	1.3825	3.8398×10^{-6}	1.3645
	0.01×2^{-4}	1.2115×10^{-6}	1.4220	1.4851×10^{-6}	1.3705
0.0050	0.01×2^{-1}	1.4692×10^{-6}	—	1.7466×10^{-6}	—
	0.01×2^{-2}	5.5316×10^{-7}	1.4093	6.6390×10^{-7}	1.3955
	0.01×2^{-3}	2.0286×10^{-7}	1.4472	2.4593×10^{-7}	1.4327
	0.01×2^{-4}	7.3657×10^{-8}	1.4616	8.8765×10^{-8}	1.4702
0.0025	0.01×2^{-1}	9.4485×10^{-8}	—	1.1457×10^{-7}	—
	0.01×2^{-2}	3.4699×10^{-8}	1.4452	4.2255×10^{-8}	1.4391
	0.01×2^{-3}	1.2550×10^{-8}	1.4672	1.5489×10^{-8}	1.4479
	0.01×2^{-4}	4.4642×10^{-9}	1.4912	5.5892×10^{-9}	1.4705

Note: The parameters and conditions employed in this case correspond to those in Example 1.. Various sets of computational parameters were employed, and two different times were used for comparisons, namely, $t = 0.5$ and $t = 1$.

convergence of the scheme. In turn, the bottom half of Table 1 provides an analysis of convergence in the temporal variable. As predicted by Theorem 3, the temporal rate of convergence is approximately equal to $\Delta^{1.5}$.

We consider a reaction–diffusion equation (1) with distributed delay in the temporal variable.

Example 2. Consider the continuous model (1) with reaction of the form

$$f(x,t,u(x,t),u_t(x,\cdot))$$
$$= \left[\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + \pi^2 t^3 - \frac{t^3}{8} - \frac{15}{64}t^4 \sin(\pi x) + \int_{-t/2}^{0} u(x,t+s)ds\right] \sin(\pi x) + u(x,t-t/2), \quad (52)$$

TABLE 2 Table of absolute errors and standard convergence rates in space when approximating the solution u of (1) and (2) with $\alpha = 0.5$, using the difference method (16) and (17)

Spatial analysis of convergence

		$t = 1.5$		$t = 2$	
Δ	h	$\epsilon_{\Delta,h}$	$\rho^x_{\Delta,h}$	$\epsilon_{\Delta,h}$	$\rho^x_{\Delta,h}$
0.04	0.01×2^{-1}	1.2846×10^{-5}	—	1.3401×10^{-5}	—
	0.01×2^{-2}	1.1572×10^{-6}	3.4726	1.2672×10^{-6}	3.4026
	0.01×2^{-3}	8.9836×10^{-8}	3.6872	1.0662×10^{-7}	3.5710
	0.01×2^{-4}	6.2711×10^{-9}	3.8405	8.4225×10^{-9}	3.6621
0.02	0.01×2^{-1}	4.9959×10^{-6}	—	5.4951×10^{-6}	—
	0.01×2^{-2}	4.0425×10^{-7}	3.6274	4.5871×10^{-7}	3.5825
	0.01×2^{-3}	3.0253×10^{-8}	3.7401	3.5159×10^{-8}	3.7056
	0.01×2^{-4}	2.0724×10^{-9}	3.8677	2.4996×10^{-9}	3.8141
0.01	0.01×2^{-1}	1.8576×10^{-6}	—	2.1284×10^{-6}	—
	0.01×2^{-2}	1.3882×10^{-7}	3.7422	1.6546×10^{-7}	3.6852
	0.01×2^{-3}	9.5774×10^{-9}	3.8574	1.1885×10^{-8}	3.7992
	0.01×2^{-4}	6.1576×10^{-10}	3.9592	8.1255×10^{-10}	3.8706

Temporal analysis of convergence

		$t = 1.5$		$t = 2$	
h	Δ	$\epsilon_{\Delta,h}$	$\rho^t_{\Delta,h}$	$\epsilon_{\Delta,h}$	$\rho^t_{\Delta,h}$
0.00500	0.02×2^{-1}	1.8576×10^{-6}	—	2.1284×10^{-6}	—
	0.02×2^{-2}	7.2254×10^{-7}	1.3623	8.1981×10^{-7}	1.3764
	0.02×2^{-3}	2.7778×10^{-7}	1.3791	3.1442×10^{-7}	1.3826
	0.02×2^{-4}	1.0524×10^{-7}	1.4003	1.1989×10^{-7}	1.3910
0.00250	0.02×2^{-1}	1.1980×10^{-7}	—	1.4093×10^{-7}	—
	0.02×2^{-2}	4.4763×10^{-8}	1.4203	5.2902×10^{-8}	1.4136
	0.02×2^{-3}	1.6239×10^{-8}	1.4628	1.9207×10^{-8}	1.4617
	0.02×2^{-4}	5.8204×10^{-9}	1.4803	6.9324×10^{-9}	1.4702
0.00125	0.02×2^{-1}	7.5151×10^{-9}	—	9.2723×10^{-9}	—
	0.02×2^{-2}	2.7625×10^{-9}	1.4438	3.3342×10^{-9}	1.4756
	0.02×2^{-3}	9.8615×10^{-10}	1.4861	1.1936×10^{-9}	1.4820
	0.02×2^{-4}	3.5101×10^{-10}	1.4903	4.2535×10^{-10}	1.4886

Note: The parameters and conditions employed in this case correspond to those in Example 2.. Various sets of computational parameters were employed, and two different times were used for comparisons, namely, $t = 1.5$ and $t = 2$.

for each $(x,t) \in [0,1] \times [1,2]$. As initial conditions, we fix $\varphi(x, t) = t^3 \sin(\pi x)$, for each $(x, t) \in [0, 1] \times \left[\frac{1}{2}, 1\right]$. The exact solution of the continuous problem (1) and (2) is given in this case by $u(x, t) = t^3 \sin(\pi x)$, for each $(x,t) \in [0,1] \times [1,2]$. Under these circumstances, Table 2 provides the spatial and temporal analysis of convergence of the numerical model when $\alpha = 0.5$. Again, the results of our simulations confirm the validity of the Theorem 3.

Finally, the following example considers a more complicated problem.

Example 3. We use the parameters of Example 1., consider the function $u(x, t) = (1 + t + t^2 + t^3 + t^4) \sin(4\pi x)$, and define f and ϕ in such way that the function

TABLE 3 Table of absolute errors and standard convergence rates in space when approximating the solution u of (1) and (2) with $\alpha = 0.5$, using the difference method (16) and (17)

Spatial analysis of convergence

		$t = 1.5$		$t = 2$	
Δ	h	$\epsilon_{\Delta,h}$	$\rho^x_{\Delta,h}$	$\epsilon_{\Delta,h}$	$\rho^x_{\Delta,h}$
0.04	0.01×2^{-1}	3.7507×10^{-8}	—	4.8915×10^{-8}	—
	0.01×2^{-2}	2.4392×10^{-9}	3.9427	3.3017×10^{-9}	3.8890
	0.01×2^{-3}	1.5358×10^{-10}	3.9893	2.1563×10^{-10}	3.9366
	0.01×2^{-4}	9.3958×10^{-12}	4.0307	1.3778×10^{-11}	3.9681
0.02	0.01×2^{-1}	1.3476×10^{-8}	—	1.8722×10^{-8}	—
	0.01×2^{-2}	8.5368×10^{-10}	3.9805	1.2024×10^{-9}	3.9608
	0.01×2^{-3}	5.0540×10^{-11}	4.0782	7.6849×10^{-11}	3.9677
	0.01×2^{-4}	3.0860×10^{-12}	4.0336	4.8674×10^{-12}	3.9808
0.01	0.01×2^{-1}	4.7388×10^{-9}	—	5.0278×10^{-9}	—
	0.01×2^{-2}	2.8270×10^{-10}	4.0672	3.1907×10^{-10}	3.9780
	0.01×2^{-3}	1.7397×10^{-11}	4.0223	1.9607×10^{-11}	4.0244
	0.01×2^{-4}	1.1011×10^{-12}	3.9815	1.2555×10^{-12}	3.9651

Temporal analysis of convergence

		$t = 1.5$		$t = 2$	
h	Δ	$\epsilon_{\Delta,h}$	$\rho^t_{\Delta,h}$	$\epsilon_{\Delta,h}$	$\rho^t_{\Delta,h}$
0.00500	0.02×2^{-1}	5.3482×10^{-9}	—	5.6982×10^{-9}	—
	0.02×2^{-2}	1.9321×10^{-9}	1.4689	2.0710×10^{-9}	1.4602
	0.02×2^{-3}	6.9229×10^{-10}	1.4807	7.4546×10^{-10}	1.4741
	0.02×2^{-4}	2.4102×10^{-10}	1.5222	2.6254×10^{-10}	1.5056
0.00250	0.02×2^{-1}	3.3930×10^{-10}	—	3.4277×10^{-10}	—
	0.02×2^{-2}	1.2077×10^{-10}	1.4903	1.2458×10^{-10}	1.4602
	0.02×2^{-3}	4.1581×10^{-11}	1.5383	4.4437×10^{-11}	1.4872
	0.02×2^{-4}	1.4032×10^{-11}	1.5672	1.6035×10^{-11}	1.4705
0.00125	0.02×2^{-1}	2.2302×10^{-11}	—	2.4900×10^{-11}	—
	0.02×2^{-2}	7.8262×10^{-12}	1.5108	8.9468×10^{-12}	1.4767
	0.02×2^{-3}	2.8306×10^{-12}	1.4672	3.2834×10^{-12}	1.4462
	0.02×2^{-4}	1.0387×10^{-12}	1.4463	1.2198×10^{-12}	1.4285

Note: The parameters and conditions employed in this case correspond to those in Example 3.. Various sets of computational parameters were employed, and two different times were used for comparisons, namely, $t = 1.5$ and $t = 2$.

u is an exact solution of the problem (1) and (2). Under these circumstances, Table 3 shows the numerical study of convergence for the current problem in both space and time. According to the results, we confirm again that the numerical solution has a convergence rate of order $\mathcal{O}(\Delta^{2-\alpha} + h^4)$ with respect to the L^∞-norm. Moreover, Tables 1–3 show that the difference scheme is independent of the number of partitions with respect to the time and space variables, for sufficiently small step sizes. This is in agreement with the results obtained Theorem 3. Moreover, this denies the possibility of the unconditional independence due to the absolute accuracy of the difference operator for the fractional derivative, as in the first two examples.

5 | CONCLUSIONS

In this work, we studied numerically a family of one-dimensional time-fractional parabolic partial differential equations with temporal delay of functional type. More precisely, we proposed a compact difference model to approximate the solutions of the mathematical model of interest. The approach hinged on the idea of separating the current state and the prehistory of the solution. Numerically, we employed an interpolation–extrapolation technique to approximate the previous history of the solution. We proved in this work that the numerical model proposed is uniquely solvable, and that it is a convergent and stable technique. The cornerstone to establish the convergence and the stability was the use of a discrete form of the fractional Gronwall inequality, which provides an optimal error estimate for the numerical solutions.

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